RANDOM MATRICES HOMEWORK SHEET 2

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To hand in by March 18 to the instructor's mail box (first floor of Schreiber building, opposite mathematics secretariat). Please write an email to inform the instructor of the submission if submitting before the due date. An extended period of time is given for the exercise in order to lighten the load during the exam period. However, early submissions are encouraged.

It is requested that solutions be written in English if possible.

The numbering of the first two exercises is from "An Introduction to Random Matrices" by Anderson, Guionnet, Zeitouni, available at `http://cims.nyu.edu/~zeitouni/cupbook.pdf`.

The numbering of the last three exercises is from "The surprising mathematics of longest increasing subsequences" by Romik, available at `http://www.math.ucdavis.edu/~romik/book/`.

(i) Solve exercise 3.7.11 from the AGZ book. In this exercise we discuss the second solution Bi (besides the Airy function Ai) to the Airy ordinary differential equation $y''(x)-x\cdot y(x)=0$. We consider in particular the Wronskian of the two solutions, as given by (3.7.29), which was used in the proof that $Ai(x)>0$ for $x>0$ in Lemma 3.7.6.

The contour in the displayed equation before (3.7.30) should be

$$(e^{-2\pi i/3}\infty,-1]+[-1,1]+[1,\infty)+(e^{2\pi i/3}\infty,-1]+[-1,1]+[1,\infty).$$

The limits in (3.7.30) and (3.7.31) are as $x\to+\infty$ over real numbers. The minus sign in front of $\pi^{-1/2}x^{1/4}e^{2/3x^{3/2}}$ in (3.7.31) should be removed.

(ii) Solve exercise 3.8.3 from the AGZ book. In this exercise we find the positive and negative tail behavior of the Tracy-Widom distribution.
Correction: The second limit should be $\lim_{t\to-\infty}\frac{1}{t^3}\log F_2(t)=+\frac{1}{12}$ (correcting the sign).

(iii) Solve exercise 1.23 from Romik's book. The exercise asks to show that in a certain 'inverse hook walk' described there, the probability starting from a Young diagram μ with $n-1$ cells, to finish in an external corner c of μ is exactly $\frac{d_{\mu\cup\{c\}}}{nd_\mu}$, where d_λ is the number of Young tableaux of shape λ.
Clarification: The exercise provides an efficient method for sampling a Young diagram from the Plancherel distribution, via the 'Plancherel growth process' which is described in Section 1.19 in Romik's book. Section 1.19 is interesting and recommended background material that is, nonetheless, not required for the solution of the exercise.
Hint: Recall the hook walk discussed in class (Section 1.9) which starts from a uniformly picked box in the Young diagram and iteratively moves to a uniformly picked box in its current hook. We proved (Proposition 1.13) that when running this walk on the Young diagram μ, the probability to end at the corner $c=(a,b)$ is $\frac{d_\mu}{d_\lambda}$, with $\lambda:=\mu\cup\{c\}$. Adapt the proof to show that if the walk is conditioned to start from the top-left corner box (having coordinates $(1,1)$), then the probability to finish at c becomes $\frac{d_\mu}{d_\lambda}\cdot\frac{n}{h_\lambda(1,b)\cdot h_\lambda(a,1)}$, where $h_\lambda(i,j)$ is the length of the (i,j) hook in λ.

Use the above formula to obtain an expression for the probability that the 'inverse hook walk' ends at a given external corner, and show that the obtained expression equals the expression required by the exercise.

(iv) Solve exercise 2.6 in Romik's book. In this exercise a formula is found for the joint probability distribution of the number of points placed by a discrete determinantal point process in prescribed disjoint sets.
Correction: The factor $N!$ should be replaced by $n_1!n_2!\cdots n_k!$.

(v) Solve exercise 2.21 in Romik's book. In this (somewhat long!) exercise an expression termed the Gessel-Bessel identity is developed for the probability distribution of the longest increasing subsequence of a random permutation (picked uniformly from S_n when n itself is first picked from the Poisson(θ) distribution). This allows an alternative (in fact, the original) approach to the Baik-Deift-Johansson theorem.

Date: March 4, 2018.